the best choice for the ratio of outside to inside pipe diameter is 1.41. Some of the heat exchangers were sized close to the optimum thermoeconomical efficiency point for the sample problem, which indicates the validity of the solution method.

The optimum pipe sizing and the most feasible operating conditions of double-pipe heat exchangers are determined by finding the point of maximum savings for typical waste heat recovery. The optimization formulation is checked by comparing the present results of a sample problem with some of the data available in double-pipe heat exchanger manufacturers catalogs. Double-pipe heat exchangers must be designed referring to this optimum point. The present formulas should to be helpful for double-pipe heat exchanger designers and manufacturers.

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Convection in a Vertical Annular Duct with Circumferentially Variable Boundary Conditions

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Nomenclature

$a_0(r)$, $a_n(r)$, $b_n(r)$	=	functions that appear in the temperature Fourier series
$c_0(r)$, $c_n(r)$, $h_n(r)$	=	functions that appear in the velocity Fourier series
F_1, F_2	=	functions that appear in Eq. (38)
f	=	Fanning friction factor
G_1, G_2	=	functions that appear in Eq. (39)
Gr	=	Grashof number
g	=	magnitude of the gravitational acceleration, m/s^2
$\boldsymbol{g}$	=	gravitational acceleration, m/s^2
k	=	thermal conductivity, W/(mK)
n, N, s, s_n	=	positive integers
P	=	difference between pressure and hydrostatic pressure, Pa
R	=	radial coordinate, m
Re	=	Reynolds number
r	=	dimensionless radial coordinate
T	=	temperature, K
T_0	=	average temperature in a duct section, K
t	=	dimensionless temperature
U	=	X component of the fluid velocity, m/s
$\boldsymbol{U}$	=	fluid velocity, m/s
U_0	=	average velocity in a duct section, m/s
u	=	dimensionless velocity
$u^{(1)}$	=	forced convection dimensionless velocity
$u^{(2)}$	=	buoyancy-induced dimensionless velocity
u^*	=	modified dimensionless velocity
X	=	axial coordinate, m
$\boldsymbol{X}$	=	unit vector parallel to the X axis
x_n, y_n	=	real numbers
β	=	volumetric coefficient of thermal expansion, K^{-1}
γ	=	aspect ratio of the duct
Δq	=	amplitude of the wall heat-flux oscillations, W/m^2
ΔT	=	reference temperature difference, K
ϑ	=	angular coordinate, rad
λ	=	dimensionless pressure drop
μ	=	dynamic viscosity, Pa s
ν	=	kinematic viscosity, m^2/s

Subscripts

e	=	external wall
FR	=	onset of flow reversal
i	=	internal wall

Introduction

SEVERAL technical papers deal with forced and free convection in annular ducts. With reference to the case of forced flow, an accurate description of the main works available in the literature is presented by Shah and London.[1] On the other hand, a wide literature exists also on mixed convection in vertical annular ducts. The analytical study of Rokerya and Iqbal[2] is among the first papers on this subject. The authors consider the fully developed flow inside a vertical annulus, taking into account also the effect of viscous dissipation. More recently, mixed convection in a vertical annular duct for a fluid with temperature-dependent properties has been studied numerically.[3,4] Mixed convection in a vertical annular duct has been studied analytically also for flows inside porous media[5] and for power law fluids.[6] Most of these studies on mixed convection refer to axisymmetric boundary conditions. However, there are several technical cases where this symmetry is not a sufficiently realistic assumption. Indeed, the duct-wall temperature and the duct-wall heat flux are often functions of the azimuthal angular coordinate. For example, absence of axial symmetry can arise in the design of heat exchangers where special flow configurations, such as crossflow, can induce a nonuniform circumferential wall temperature distribution. Similarly, thermal boundary conditions that depend on the angular coordinate can occur in the thermal control of ducts for transport of water, gas, or hydrocarbons, which are partially buried either in the soil or in a wall. With reference to circular ducts, nonaxisymmetric thermal boundary conditions have been studied by Reynolds[7] and, more recently, by Choi and Choi.[8] Sutherland and Kays[9] have studied the forced convection regime for either laminar or turbulent flow in an annular duct. These authors have considered heat-flux distributions at the walls, which are not axisymmetric and which are uniform in the axial direction.

In the present Note, the laminar mixed convection in a vertical annular duct is studied. The fully developed region is considered

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under the hypothesis of parallel flow. The momentum local-balance equation and the energy local-balance equation are solved analytically by means of the Fourier-series expansions method. Moreover, the Fanning friction factors are determined. Two particular cases are considered: an annular duct with the inner wall adiabatic and the outer wall subjected to a sinusoidal heat-flux distribution in the angular direction; an annulus with the inner wall subjected to a sinusoidal heat-flux distribution and the outer wall adiabatic. The threshold value of the ratio Gr/Re for the onset of the phenomenon of flow reversal is determined as a function of the duct aspect ratio. Finally, the solution for a more general case is determined: the annular duct has both walls subjected to arbitrary heat-flux distributions, which can be described as Fourier sine series with respect to the angular coordinate.

Governing Equations

Let us consider the steady and fully developed laminar flow of a Newtonian fluid in a vertical annular duct with inner radius R_i and outer radius R_e. The duct and the cylindrical coordinate system (X, R, ϑ) adopted are sketched in Fig. 1. A parallel flow is considered, so that the velocity vector $\boldsymbol{U}$ is directed along the X axis, that is, $\boldsymbol{U} = U\boldsymbol{X}$. Moreover, the effect of viscous dissipation is neglected in this study. The Boussinesq approximation is employed, so that $\boldsymbol{U}$ is a solenoidal field, and, as a consequence, $\partial U/\partial X = 0$. The thermal boundary conditions are such that no net fluid heating occurs in the axial direction, that is, $\partial T/\partial X = 0$. Under these assumptions, one can easily prove that the difference between the pressure and the hydrostatic pressure depends only on X, that is, $P = P(X)$.

The average fluid temperature on a duct cross section

$$T_0 = \frac{1}{\pi\left(R_e^2 - R_i^2\right)} \int_0^{2\pi} \mathrm{d}\vartheta \int_{R_i}^{R_e} \mathrm{d}R\, R\, T(R, \vartheta) \qquad (1)$$

is considered as the reference fluid temperature in applying the Boussinesq approximation.[10] Obviously, T_0 is a constant. Let us define the dimensionless quantities

$$t = \frac{T - T_0}{\Delta T}, \qquad u = \frac{U}{U_0}$$

$$r = \frac{R}{R_e}, \qquad \gamma = \frac{R_i}{R_e}, \qquad \lambda = -\frac{4(R_e - R_i)^2}{\mu U_0}\frac{\mathrm{d}P}{\mathrm{d}X}$$

$$Re = \frac{2(R_e - R_i)U_0}{\nu}, \qquad Gr = \frac{8g\beta\Delta T(R_e - R_i)^3}{\nu^2} \qquad (2)$$

where U_0 is given by

$$U_0 = \frac{1}{\pi\left(R_e^2 - R_i^2\right)} \int_0^{2\pi} \mathrm{d}\vartheta \int_{R_i}^{R_e} \mathrm{d}R\, R\, U(R, \vartheta) \qquad (3)$$

Obviously, U_0 is a constant. In the following, ΔT is assumed to be a positive quantity, so that positive values of Gr/Re imply positive values of U_0 (upward flow), while negative values of Gr/Re imply negative values of U_0 (downward flow). This quantity is fixed once the thermal boundary conditions are prescribed. Definitions of ΔT are given in the next sections, with reference to a pair of specific examples.

By means of Eq. (2), the momentum-balance equation along X and the energy-balance equation can be written in a dimensionless form as

$$\frac{Gr}{Re}t + \lambda + 4(1-\gamma)^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2}\right) = 0 \qquad (4)$$

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \vartheta^2} = 0 \qquad (5)$$

The no-slip condition at the walls implies that

$$u(\gamma, \vartheta) = 0 = u(1, \vartheta) \qquad (6)$$

Finally, Eqs. (1) and (3) imply the following constraints on the functions $t(r, \vartheta)$ and $u(r, \vartheta)$:

$$\int_0^{2\pi} \mathrm{d}\vartheta \int_\gamma^1 \mathrm{d}r\, t\, r = 0 \qquad (7)$$

$$\int_0^{2\pi} \mathrm{d}\vartheta \int_\gamma^1 \mathrm{d}r\, u\, r = \pi(1 - \gamma^2) \qquad (8)$$

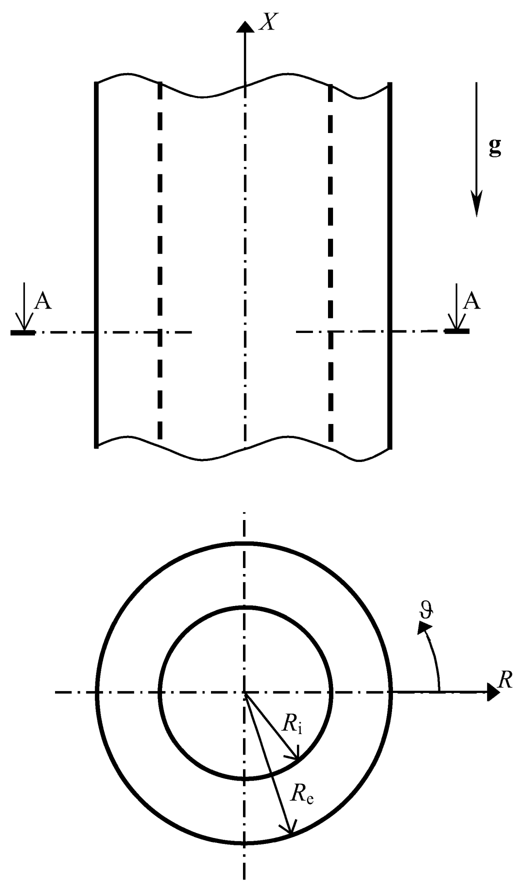

Fig. 1 Drawing of the annular duct.

Dimensionless Temperature and Velocity Fields

The dimensionless temperature $t(r, \vartheta)$ is continuous for $\gamma \le r < 1$ and $0 \le \vartheta < 2\pi$, so that it can be expanded as a Fourier series with respect to the variable ϑ, as follows:

$$t(r, \vartheta) = \frac{a_0(r)}{2} + \sum_{n=1}^{\infty} [a_n(r)\sin(n\vartheta) + b_n(r)\cos(n\vartheta)] \qquad (9)$$

The coefficients $a_n(r)$ and $b_n(r)$ can be obtained by substituting Eq. (9) in Eq. (5) and by employing the orthogonality relations of trigonometric functions as well as the constraint given by Eq. (7). On account of Eq. (9), Eq. (7) acts as a constraint only on $a_0(r)$.

To obtain the dimensionless velocity distribution, a procedure similar to that employed for the dimensionless temperature distribution can be followed. In fact, the function $u(r, \vartheta)$ is continuous for $\gamma \le r < 1$ and $0 \le \vartheta < 2\pi$, so that it can be expanded as a Fourier series with respect to the variable ϑ, as follows:

$$u(r, \vartheta) = \frac{c_0(r)}{2} + \sum_{n=1}^{\infty}[c_n(r)\sin(n\vartheta) + h_n(r)\cos(n\vartheta)] \qquad (10)$$

Equation (8) can be used to determine the value of the parameter λ. If the thermal boundary conditions are such that, for every X

location, the average heat flux is zero on each of the boundaries $r=\gamma$ and $r=1$, then $a_0(r)=0$. In this special case, the parameter λ is given by

$$\lambda = \frac{32(1-\gamma)^2 \ln\gamma}{1-\gamma^2+(1+\gamma^2)\ln\gamma} \tag{11}$$

Functions $c_0(r)$, $c_n(r)$, and $h_n(r)$ can be determined by employing the orthogonality relations of trigonometric functions and the local momentum-balance equation (4). On account of Eq. (10), it can be proved that the dimensionless velocity distribution can be expressed as

$$u(r,\vartheta) = u^{(1)}(r) + (Gr/Re)u^{(2)}(r,\vartheta) \tag{12}$$

where

$$u^{(1)}(r) = \frac{2\left[(1-r^2)\ln\gamma - (1-\gamma^2)\ln r\right]}{1-\gamma^2+(1+\gamma^2)\ln\gamma} \tag{13}$$

and $u^{(2)}(r,\vartheta)$ can be determined only once the thermal boundary conditions have been prescribed.

On account of Eq. (12), in the limit $Gr/Re \to 0$, $u(r,\vartheta)$ tends to $u^{(1)}(r)$, that is, the well-known dimensionless velocity distribution for isothermal flow. Indeed, the limit $Gr/Re \to 0$ represents the forced convection regime. Moreover, if one defines a modified dimensionless velocity through the expression

$$u^*(r,\vartheta) = \frac{Re}{Gr}u(r,\vartheta) = \frac{\nu U(R,\vartheta)}{4g\beta\Delta T(R_e-R_i)^2} \tag{14}$$

$u^{(2)}(r,\vartheta)$ coincides with the limit of $u^*(r,\vartheta)$ for $Gr/Re \to \infty$. This limit corresponds to the case of free convection (purely buoyancy–driven flow).

With reference to the inner and the outer wall, the following Fanning friction factors can be obtained:

$$f_i Re = \frac{2(1-\gamma)}{\pi}\int_0^{2\pi} \left.\frac{\partial u}{\partial r}\right|_{r=\gamma} d\vartheta \tag{15}$$

$$f_e Re = -\frac{2(1-\gamma)}{\pi}\int_0^{2\pi} \left.\frac{\partial u}{\partial r}\right|_{r=1} d\vartheta \tag{16}$$

If one integrates both sides of Eq. (4) with respect to r and ϑ on the whole annular cross section, one is led to the expression:

$$\lambda\pi(1+\gamma)+4(1-\gamma) \times \left(-\gamma\int_0^{2\pi}\left.\frac{\partial u}{\partial r}\right|_{r=\gamma} d\vartheta + \int_0^{2\pi}\left.\frac{\partial u}{\partial r}\right|_{r=1} d\vartheta\right) = 0 \tag{17}$$

Equations (15–17) yield the relation

$$(f_i\gamma + f_e)Re = (\lambda/2)(1+\gamma) \tag{18}$$

In the following sections, the general solution just obtained will be applied to a couple of fundamental cases. In particular, these cases refer to thermal boundary conditions in which one of the walls is adiabatic and the other one is subjected to a sinusoidal heat-flux distribution. Then, the solution of the more general case in which both walls are subjected to arbitrary heat-flux distributions that can be described as Fourier sine series is discussed. The solution in this general case is expressed as a superposition of the two fundamental solutions just described.

First Example

Let us consider the case of an annular duct with the inner wall adiabatic and the outer wall subjected to a heat-flux distribution that is sinusoidal with respect to the angular coordinate ϑ. The thermal boundary conditions can be expressed as

$$\left.\frac{\partial T}{\partial R}\right|_{R=R_i} = 0, \qquad \left.\frac{\partial T}{\partial R}\right|_{R=R_e} = \sin(s\vartheta)\frac{\Delta q}{k} \tag{19}$$

where s is an arbitrary positive integer that determines the frequency of the sinusoidal wall heat-flux distribution. By choosing the reference temperature difference ΔT as

$$\Delta T = R_e\Delta q/k \tag{20}$$

Eqs. (2) and (19) yield

$$\left.\frac{\partial t}{\partial r}\right|_{r=\gamma} = 0, \qquad \left.\frac{\partial t}{\partial r}\right|_{r=1} = \sin(s\vartheta) \tag{21}$$

By means of Eq. (21), one obtains $a_0(r)=0$. Therefore, one can conclude that, in this case, λ is given by Eq. (11), and, hence, it is not influenced by buoyancy. Moreover, the dimensionless temperature distribution is given by

$$t(r,\vartheta) = \frac{r^s+\gamma^{2s}r^{-s}}{s(1-\gamma^{2s})}\sin(s\vartheta) \tag{22}$$

whereas the buoyancy-induced dimensionless velocity term $u^{(2)}$ is given by

$$u^{(2)}(r,\vartheta) = \{(1-r^2)[(r^2-\gamma^2)(1-\gamma^2)+4\gamma^4\ln\gamma] - 4r^2\gamma^2(1-\gamma^2)\ln r\}[32r(1-\gamma)^4(1+\gamma)^2]^{-1}\sin(\vartheta) \quad \text{for} \quad s=1 \tag{23}$$

$$u^{(2)}(r,\vartheta) = \{r^{2s}(1-r^2)(s-1)+\gamma^{4s}(1-r^2)(s+1)+\gamma^{2s} \times [1-s+r^2(1+s)-2\gamma^2-r^{2s}+r^{2(s+1)}(s-1)-sr^{2s} + 2(r^s\gamma)^2]\}[16r^s s(s^2-1)(1-\gamma)^2(1-\gamma^{2s})^2]^{-1}\sin(s\vartheta) \quad \text{for} \quad s>1 \tag{24}$$

On account of Eqs. (12), (13), (15), (16), (23), and (24), the following expressions of the Fanning friction factors can be obtained:

$$f_i Re = -\frac{8(1-\gamma)(1-\gamma^2+2\gamma^2\ln\gamma)}{\gamma[1-\gamma^2+(1+\gamma^2)\ln\gamma]} \tag{25}$$

$$f_e Re = \frac{8(1-\gamma)(1-\gamma^2+2\ln\gamma)}{1-\gamma^2+(1+\gamma^2)\ln\gamma} \tag{26}$$

Both the expression of $f_i Re$ and that of $f_e Re$ are independent of s and Gr/Re, that is, they coincide with the expressions for the case of forced convection. It can be observed that for every s there exists a positive real number $(Gr/Re)_{\mathrm{FR}}$ such that flow reversal occurs for upward flow when $Gr/Re > (Gr/Re)_{\mathrm{FR}}$ and for downward flow when $Gr/Re < -(Gr/Re)_{\mathrm{FR}}$. For upward flow, the onset of flow reversal occurs in the coolest point of the boundary, namely, for $r=1$ and $\vartheta = 3\pi/(2s)$. Indeed, for $Gr/Re = (Gr/Re)_{\mathrm{FR}}$, the first derivative of $u(r,\vartheta)$, evaluated for $r=1$ and $\vartheta=3\pi/(2s)$, vanishes so that

$$(Gr/Re)_{\mathrm{FR}} = [32(1-\gamma)^4(\gamma+1)^2(1-\gamma^2+2\ln\gamma)] \times \{(1-\gamma^4+4\gamma^4\ln\gamma)[1-\gamma^2+(1+\gamma^2)\ln\gamma]\}^{-1} \quad \text{for} \quad s=1 \tag{27}$$

$$(Gr/Re)_{\mathrm{FR}} = [16s(s^2-1)(1-\gamma)^2(1-\gamma^{2s})^2(1-\gamma^2+2\ln\gamma)] \times \left\{\left[s-1+(s+1)\gamma^{4s}-2s\gamma^{2(1+s)}\right] \times [1-\gamma^2+(1+\gamma^2)\ln\gamma]\right\}^{-1} \quad \text{for} \quad s>1 \tag{28}$$

In Fig. 2, the dimensionless velocity distribution $u(r,\vartheta)$ is plotted for $\gamma=0.5$, $s=1$, and $Gr/Re=100$. This figure shows that strong deformations of the Poiseuille dimensionless velocity distribution occur. Namely, high positive values of u take place in the hotter part

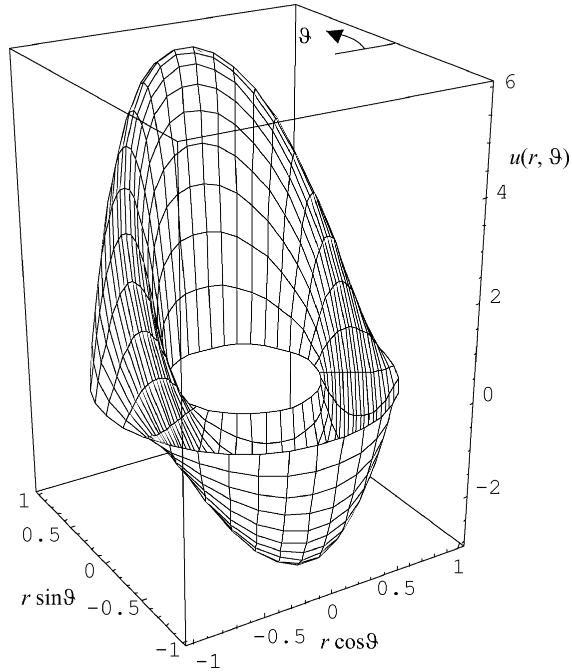

Fig. 2 Example 1: dimensionless velocity distribution $u(r, \vartheta)$, for $\gamma = 0.5$, $s = 1$, and $Gr/Re = 100$.

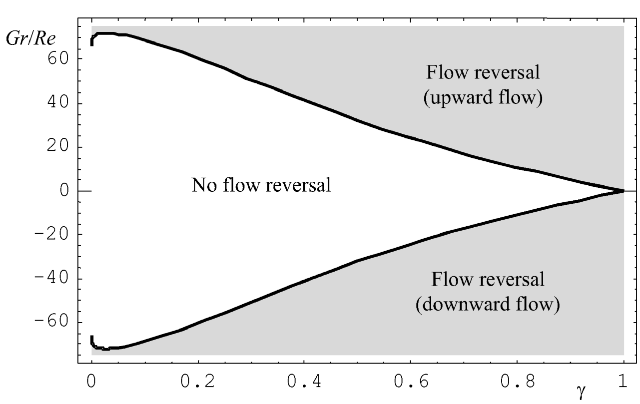

Fig. 3 Example 1: flow reversal regions for $s = 1$.

of the duct, whereas negative values of u are present close to the cooler part of the wall (flow reversal).

In Fig. 3, the regions in the $(\gamma, Gr/Re)$ plane corresponding to flow reversal are represented for $s = 1$. This figure shows that the regions where flow reversal phenomena arise, either for upward flow or for downward flow, become wider as γ increases. Indeed, as the section available for the flow decreases, the influence of the thermal boundary conditions on the velocity field becomes more important. Moreover, for $s = 1$ it can be easily verified that the threshold value $(Gr/Re)_{\mathrm{FR}}$ tends to 64 in the limit $\gamma \to 0$.

Second Example

Let us consider a duct with an adiabatic outer wall and the inner wall subjected to a heat flux varying sinusoidally with respect to the angular coordinate ϑ. The thermal boundary conditions can be expressed as

$$-\left.\frac{\partial T}{\partial R}\right|_{R=R_i} = \sin(s\vartheta)\frac{\Delta q}{k}, \qquad \left.\frac{\partial T}{\partial R}\right|_{R=R_e} = 0 \tag{29}$$

By choosing the reference temperature difference ΔT as

$$\Delta T = R_i \Delta q / k \tag{30}$$

Eqs. (2) and (29) yield

$$-\left.\frac{\partial t}{\partial r}\right|_{r=\gamma} = \sin(s\vartheta), \qquad \left.\frac{\partial t}{\partial r}\right|_{r=1} = 0 \tag{31}$$

By means of Eq. (31), also in this case one obtains $a_0(r) = 0$. Therefore, one can conclude, as in the preceding example, that λ is not influenced by buoyancy and is given by Eq. (11). The dimensionless temperature distribution can be expressed as

$$t(r, \vartheta) = \frac{(r^s + r^{-s})\gamma^{1+s}}{s(1-\gamma^{2s})}\sin(s\vartheta) \tag{32}$$

whereas the buoyancy-induced dimensionless velocity term $u^{(2)}$ is given by

$$u^{(2)}(r, \vartheta) = \gamma^2\{(1-\gamma^2)[(1-r^2)(r^2-\gamma^2) - 4r^2 \ln r] + 4(1-r^2)\gamma^2 \ln\gamma\}[32r(1-\gamma)^4(1+\gamma)^2]^{-1}\sin(\vartheta) \quad \text{for} \quad s = 1 \tag{33}$$

$$u^{(2)}(r, \vartheta) = \gamma^{1+s}\{\gamma^{2s}[2+(s-1)\gamma^2] - (1+s)\gamma^2 - r^{2(1+s)} \times (s-1)(1-\gamma^{2s}) + r^2(s+1)(1-\gamma^{2s}) - 2r^{2s} + (r^s\gamma)^2 \times [1+s-(s-1)\gamma^{2s}]\}[16r^s s(s^2-1) \times (1-\gamma)^2(1-\gamma^{2s})^2]^{-1}\sin(s\vartheta) \quad \text{for} \quad s > 1 \tag{34}$$

As can be easily verified by means of Eqs. (12), (13), (15), (16), (33), and (34), the expressions of $f_i Re$ and $f_e Re$ coincide with those given by Eqs. (25) and (26). By a treatment similar to the one employed in the preceding section, the threshold value of Gr/Re for the onset of flow reversal can be obtained as

$$(Gr/Re)_{\mathrm{FR}} = [32(1-\gamma)^4(1+\gamma)^2(1-\gamma^2+2\gamma^2\ln\gamma)] \times \{\gamma^3(1-\gamma^4+4\ln\gamma)[1-\gamma^2+(1+\gamma^2)\ln\gamma]\}^{-1} \quad \text{for} \quad s = 1 \tag{35}$$

$$(Gr/Re)_{\mathrm{FR}} = [16s(s^2-1)(1-\gamma)^2(1-\gamma^{2s})^2(1-\gamma^2+2\gamma^2\ln\gamma)] \times \left\{\left[2s\gamma^{2s} - (s+1)\gamma^2 - (s-1)\gamma^{2(1+2s)}\right] \times \gamma[1-\gamma^2+(1+\gamma^2)\ln\gamma]\right\}^{-1} \quad \text{for} \quad s > 1 \tag{36}$$

In Fig. 4, the dimensionless velocity distribution $u(r, \vartheta)$ is plotted for $\gamma = 0.5$, $s = 1$, and $Gr/Re = 100$. Because this value of Gr/Re is greater than the value of $(Gr/Re)_{\mathrm{FR}}$ for $s = 1$, as can be easily verified by means of Eq. (35), flow reversal phenomena close to the cooler part of the inner wall occur.

In Fig. 5, a representation of the regions in the $(\gamma, Gr/Re)$ plane that correspond to flow reversal is given for $s = 1$. This figure shows that, in the limit $\gamma \to 0$, the threshold value $(Gr/Re)_{\mathrm{FR}}$ tends to infinity, that is, flow reversal never occurs. In fact, in this limiting case the area of the internal heating surface tends to zero, and buoyancy forces vanish.

Discussion of a More General Thermal Boundary Condition

Let us consider the case of an annular duct with both walls subjected to nonaxisymmetric heat-flux distributions such that the average heat flux on each wall vanishes. In particular, let the dimensionless thermal boundary conditions be given by

$$-\left.\frac{\partial t}{\partial r}\right|_{r=\gamma} = \sum_{n=1}^{N} x_n \sin(s_n\vartheta), \qquad \left.\frac{\partial t}{\partial r}\right|_{r=1} = \sum_{n=1}^{N} y_n \sin(s_n\vartheta) \tag{37}$$

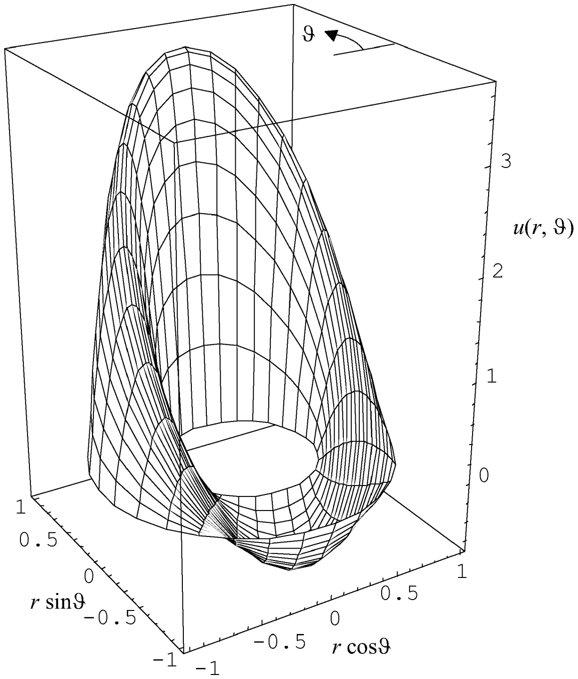

Fig. 4 Example 2: dimensionless velocity distribution $u(r, \vartheta)$, for $\gamma = 0.5$, $s = 1$, and $Gr/Re = 100$.

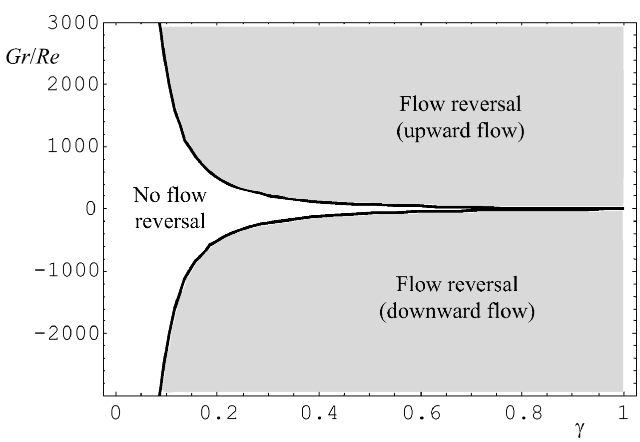

Fig. 5 Example 2: flow reversal regions for $s = 1$.

where x_n and y_n are arbitrary real numbers and s_n and N are arbitrary positive integers. In the limit $N \to \infty$, Eq. (37) corresponds to wall heat fluxes expressed as Fourier sine series with respect to the angular coordinate.

It can be easily proved that the dimensionless temperature distribution is given by the following expression:

$$t(r, \vartheta) = \sum_{n=1}^{N} [x_n F_1(r, \vartheta, s_n, \gamma) + y_n F_2(r, \vartheta, s_n, \gamma)] \quad (38)$$

where $F_1(r, \vartheta, s, \gamma)$ and $F_2(r, \vartheta, s, \gamma)$ coincide respectively with the right-hand side of Eq. (32) and with the right-hand side of Eq. (22).

The dimensionless velocity distribution is

$$u(r, \vartheta) = u^{(1)}(r) + \frac{Gr}{Re} \sum_{n=1}^{N} [x_n G_1(r, \vartheta, s_n, \gamma) + y_n G_2(r, \vartheta, s_n, \gamma)] \quad (39)$$

where $u^{(1)}(r)$ is given by Eq. (13). Function $G_1(r, \vartheta, s, \gamma)$ coincides with the right-hand side of Eq. (33) or of Eq. (34), depending on the values of s. Function $G_2(r, \vartheta, s, \gamma)$ coincides with the right-hand side of Eq. (23) or of Eq. (24), depending on the values of s. Moreover, the dimensionless pressure drop parameter λ is independent of Gr/Re and is given by Eq. (11).

Conclusions

The laminar mixed convection of a Newtonian fluid in a vertical annular duct has been analyzed under conditions of fully developed regime and parallel flow. The Boussinesq approximation has been employed. Nonaxisymmetric thermal boundary conditions have been considered. The solution of the momentum- and energy-balance equations has been found analytically by utilizing the Fourier-series expansion method. Moreover, the dimensionless pressure drop parameter and the Fanning friction factors have been determined. Two basic cases have been analyzed: in the first case, the inner wall of the duct is adiabatic, and the outer wall is subjected to a sinusoidal heat-flux distribution; in the second case, the outer wall of the duct is adiabatic, and the inner wall is subjected to a sinusoidal heat-flux distribution. Then, a more general case such that both walls are subjected to arbitrary heat-flux distributions, which can be described as Fourier sine series, has been analyzed. The main results obtained are the following:

1) The expressions of the Fanning friction factors are the same for both basic cases, and they coincide with the expressions for the case of forced convection.

2) The dimensionless pressure drop parameter is independent of buoyancy whenever the wall heat fluxes can be expressed as Fourier sine series with respect to the angular coordinate.

3) For given values of the duct aspect ratio and of the frequency of the sinusoidal wall heat-flux distribution, there exists a pair of threshold values of the ratio between the Grashof number and the Reynolds number for the onset of flow reversal: one for upward flow and the other for downward flow. These threshold values have been found analytically.

The solution obtained can become unstable, for any fixed value of the Reynolds number Re, when the buoyancy parameter Gr/Re exceeds a critical value. This critical value can be evaluated through a stability analysis.

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